Transformada seno - Transformada coseno

Fórmula integral de Fourier:

$$\boxed{f^*(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\,d\omega}$$ con: $f^*(x) = \frac{f(x^-)+f(x^+)}{2}$

$$f^*(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt\right)e^{i\omega x}\,d\omega$$

Si f es impar: $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt - i\int_{-\infty}^{\infty} f(t)\,\mathrm{sen}(\omega t)\,dt$

$$\hat{f}(\omega) = -i\int_{-\infty}^{\infty} f(t)\,\mathrm{sen}(\omega t)\,dt$$

$$\hat{f}(\omega) = -i\,2\int_{0}^{\infty} f(t)\,\mathrm{sen}(\omega t)\,dt \rightarrow \text{y es una función impar de } \omega$$

$$\hat{f}(\omega) = -\hat{f}(-\omega)$$

La fórmula integral de Fourier resulta:

$$f^*(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} -i2\int_{0}^{\infty} f(t)\,\mathrm{sen}(\omega t)\,dt \cdot e^{i\omega x}\,d\omega$$

$$= \frac{-i}{\pi}\int_{-\infty}^{\infty}\left(\int_{0}^{\infty} f(t)\,\mathrm{sen}(\omega t)\,dt\right)[\cos(\omega x) + i\,\mathrm{sen}(\omega x)]\,d\omega$$

$$= \frac{-i}{\pi}\int_{-\infty}^{\infty}\left(\int_{0}^{\infty} f(t)\,\mathrm{sen}(\omega t)\,dt\right)\cos(\omega x)\,d\omega + \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(t)\,\mathrm{sen}(\omega t)\,dt\,\mathrm{sen}(\omega x)\,d\omega$$

$$f^*(x) = \frac{2}{\pi}\int_{0}^{\infty}\underbrace{\int_{0}^{\infty} f(t)\,\mathrm{sen}(\omega t)\,dt}_{\mathcal{F}_s(f)(\omega)} \cdot \mathrm{sen}(\omega x)\,d\omega$$

Ahora: dada $f: [0,\infty) \rightarrow \mathbb{R}$, ~~con f y f' continua por tramos~~,

Transformada seno de f: $\mathcal{F}_s(f)(\omega) = \int_0^{\infty} f(t)\,\mathrm{sen}(\omega t)\,dt$

Similarmente se puede obtener: (¡hacerlo!)

si f es par:

$$f^*(x) = \frac{2}{\pi}\int_0^{\infty} \underbrace{\left(\int_0^{\infty} f(t)\cos(\omega t)\,dt\right)}_{\mathcal{F}_c(f)(\omega)} \cos(\omega x)\,d\omega$$

Dada $f:[0,\infty)\to\mathbb{R}$,

Transformada coseno: $\tilde{\mathcal{F}}_c(f)(\omega) = \int_0^{\infty} f(t)\cos(\omega t)\,dt$

Siendo $f:[0,\infty)\to\mathbb{R}$:

- $2i\,\mathcal{F}_s(f)$: transformada de Fourier de la extensión impar de f.
- $2\,\mathcal{F}_c(f)$: transformada de Fourier de la extensión par de f.

Propiedad $f:[0,\infty)\to\mathbb{R}$

$$\tilde{\mathcal{F}}_s(f')(\omega) = \int_0^{\infty} f'(t)\,\mathrm{sen}(\omega t)\,dt = f(t)\,\mathrm{sen}(\omega t)\Big|_0^{\infty} - \int_0^{\infty} f(t)\cdot\omega\cdot\cos(\omega t)\,dt$$

$$\underset{\text{si } \lim_{t\to\infty} f(t)=0}{=} -\omega\int_0^{\infty} f(t)\cos(\omega t)\,dt = -\omega\,\tilde{\mathcal{F}}_c(f)(\omega)$$

$$\tilde{\mathcal{F}}_c(f')(\omega) = \int_0^{\infty} f'(t)\cos(\omega t)\,dt = f(t)\cos(\omega t)\Big|_0^{\infty} - \int_0^{\infty} f(t)(-\omega)\,\mathrm{sen}(\omega t)\,dt$$

$$\underset{\text{si } \lim_{t\to\infty} f(t)=0}{=} -f(0) + \omega\int_0^{\infty} f(t)\,\mathrm{sen}(\omega t)\,dt = -f(0) + \omega\,\tilde{\mathcal{F}}_s(f)(\omega)$$

Si f y f' son continuas en $[0,\infty)$ y $f \xrightarrow[t\to\infty]{} 0$, f absolutamente integrable en $[0,\infty)$, entonces:

$$\boxed{\begin{aligned} \tilde{\mathcal{F}}_s(f')(\omega) &= -\omega\,\tilde{\mathcal{F}}_c(f)(\omega) \\ \tilde{\mathcal{F}}_c(f')(\omega) &= \omega\,\tilde{\mathcal{F}}_s(f)(\omega) - f(0) \end{aligned}}$$

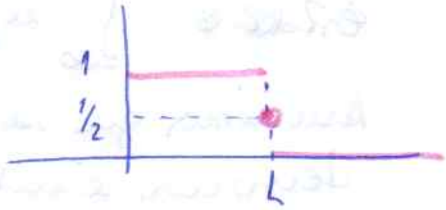
Se extiende:

$$\widetilde{\mathcal{F}}_s(f'')(\omega) = -\omega\, \widetilde{\mathcal{F}}_c(f')(\omega) = -\omega\left[\omega\, \widetilde{\mathcal{F}}_s(f)(\omega) - f(0)\right] = -\omega^2 \widetilde{\mathcal{F}}_s(f)(\omega) + \omega f(0)$$

$$\widetilde{\mathcal{F}}_c(f'')(\omega) = \omega\, \widetilde{\mathcal{F}}_s(f')(\omega) - f'(0) = \omega\left[-\omega\, \widetilde{\mathcal{F}}_c(f)(\omega)\right] - f'(0) = -\omega^2 \widetilde{\mathcal{F}}_c(f)(\omega) - f'(0)$$

$$\boxed{\begin{aligned} \widetilde{\mathcal{F}}_s(f'')(\omega) &= -\omega^2 \widetilde{\mathcal{F}}_s(f)(\omega) + \omega f(0) \\ \widetilde{\mathcal{F}}_c(f'')(\omega) &= -\omega^2 \widetilde{\mathcal{F}}_c(f)(\omega) - f'(0) \end{aligned}}$$

<u>Ejemplo</u>. Sea $f(t) = \begin{cases} 1 & 0<t<L \\ 0 & t>L \\ 1/2 & t=L \end{cases}$



Verificar: $f(t) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(L\omega)}{\omega}\,\mathrm{sen}(\omega t)\, d\omega$

Calculemos $\widetilde{\mathcal{F}}_s(f)(\omega) = \int_0^\infty f(t)\,\mathrm{sen}(\omega t)\, dt = \int_0^L 1\,\mathrm{sen}(\omega t)\, dt$

$$= -\frac{\cos(\omega t)}{\omega}\Big|_0^L = \frac{-\cos(\omega L)+1}{\omega} = -2i\mathcal{F}(\tilde{f})(\omega)$$

$\tilde{f}$: extensión impar de f.

Como f y f' son continuas por tramos,
f absolutamente integrable en $[0,\infty)$ $\Rightarrow$ vale la fórmula integral de Fourier:

$$f(t) = \frac{2}{\pi}\int_0^\infty \widetilde{\mathcal{F}}_s(f)(\omega)\,.\,\mathrm{sen}(\omega t)\, dt = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(\omega L)}{\omega}\,\mathrm{sen}(\omega t)\, d\omega$$

para $t>0$

(observar que en cada punto $t>0$ $f(t) = \frac{f(t^-)+f(t^+)}{2}$)

Aplicaciones

Ejemplo 1

Considerar:

$$\boxed{\begin{array}{ll} k\, u''_{xx}(x,t) = u'_t(x,t) & -\infty<x<\infty,\ t>0 \\ u(x,0) = f(x) & -\infty<x<\infty \end{array}}$$

Suponemos $f \in L^1(\mathbb{R})$ y para cada t, $u(x,t) \in L^1(\mathbb{R})$

Transformamos: $\left\{ \hat{U}(\omega,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx \right\}$

~~Entonces~~ $\int_{-\infty}^{\infty} u'_t(x,t) e^{-i\omega x} dx = \mathcal{F}(u'_t)(\omega,t)$

Asumimos que se verifican las hipótesis para intercambio derivación e integración:

$$\mathcal{F}(u'_t)(\omega,t) = \frac{d}{dt}\int_{-\infty}^{\infty} u(x,t) e^{-i\omega x} dx = \frac{d}{dt}\hat{U}(\omega,t) = \hat{U}'_t(\omega,t)$$

Además: $\mathcal{F}(u''_{xx})(\omega,t) = (i\omega)^2\, \mathcal{F}(u)(\omega,t) = -\omega^2\, \hat{U}(\omega,t)$

La E.D. resulta:

$$k(-\omega^2)\hat{U}(\omega,t) = \hat{U}'_t(\omega,t) \qquad \rightarrow \text{EDO 1º orden en } t$$

$$\Rightarrow \quad \hat{U}(\omega,t) = A(\omega)\cdot e^{-k\omega^2 t}$$

En $t=0$: $\hat{U}(\omega,0) = A(\omega) = \int_{-\infty}^{\infty} u(x,0) e^{-i\omega x} dx = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$

$$A(\omega) = \hat{f}(\omega)$$

$$\Rightarrow \boxed{\hat{U}(\omega,t) = \hat{f}(\omega)\cdot e^{-k\omega^2 t}}$$

como $e^{-k\omega^2 t} = \hat{g}(\omega,t)$ con $g(x,t) = \sqrt{\dfrac{1}{4tk\pi}}\, e^{-\frac{x^2}{4tk}}$

$\downarrow$

$$\exp\left(\frac{-\omega^2}{4\cdot\frac{1}{4tk}}\right)$$

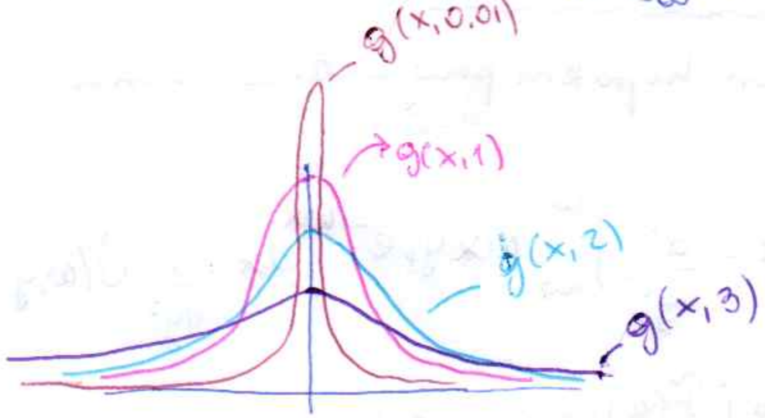
$$u(x,t) = f(x) * g(x,t) = \int_{-\infty}^{\infty} f(z)\, g(x-z,t)\, dz$$

$$= \int_{-\infty}^{\infty} f(z)\, e^{-\frac{(x-z)^2}{4tk}} \cdot \frac{1}{\sqrt{4tk\pi}}\, dz$$

$$\boxed{= \int_{-\infty}^{\infty} f(x-z)\, e^{-\frac{z^2}{4tk}} \cdot \frac{1}{\sqrt{4tk\pi}}\, dz}$$

(Otra opción: $u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}(\omega,t)\, e^{i\omega x}\, d\omega$

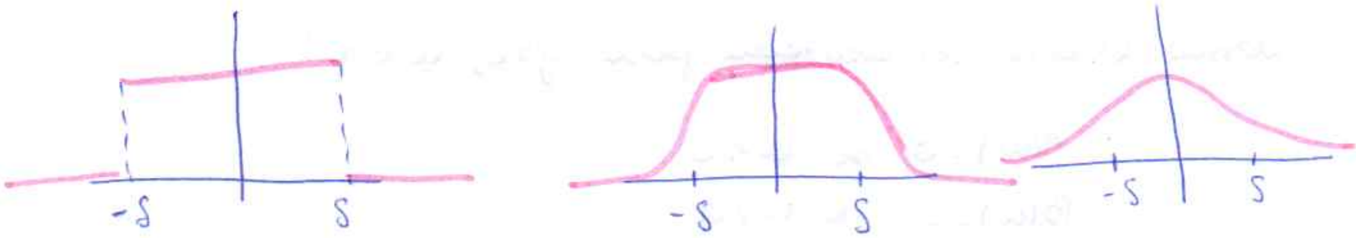
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-k\omega^2 t} \cdot e^{i\omega x}\, d\omega \quad)$$



$g(x,t)$: función de densidad de distribución normal con media 0 y varianza creciente con t.

Por ejemplo, si $f(x) = \begin{cases} 1 & \text{si } |x| < \delta \\ 0 & \text{si } |x| > \delta \end{cases}$

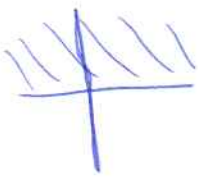
$$u(x,t) = \int_{-\delta}^{\delta} e^{-\frac{(x-z)^2}{4tk}} \frac{1}{\sqrt{4tk\pi}}\, dz = \int_{x-\delta}^{x+\delta} e^{-\frac{z^2}{4tk}} \frac{1}{\sqrt{4tk\pi}}\, dz$$



Ejemplo 2

Considerar

$$\boxed{\begin{array}{ll} u''_{xx}(x,y) + u''_{yy}(x,y) = 0 & x \in \mathbb{R},\ y>0 \\ u(x,0) = f(x) & x \in \mathbb{R}. \\ u \text{ acotada para } y>0. & \end{array}}$$



Suponemos $f \in L^1(\mathbb{R})$, $u(x,y) \in L^1(\mathbb{R})$ para cada $y>0$

Transformamos:

$$\hat{U}(\omega,y) = \int_{-\infty}^{\infty} u(x,y)\, e^{-i\omega x}\, dx$$

Asumimos que se verifican hipótesis para intercambiar derivada con integral:

$$\int_{-\infty}^{\infty} u''_{yy}(x,y)\, e^{-i\omega x}\, dx = \frac{d^2}{dy^2}\int_{-\infty}^{\infty} u(x,y)\, e^{-i\omega x}\, dx = \frac{d^2}{dy^2}\hat{U}(\omega,y)$$

Además: $\tilde{\mathcal{F}}(u''_{xx})(\omega,y) = (i\omega)^2\, \tilde{\mathcal{F}}(u)(\omega,y) = -\omega^2\, \hat{U}(\omega,y)$

La E.D. resulta:

$$-\omega^2 \hat{U}(\omega,y) + \hat{U}''_{yy}(\omega,y) = 0 \qquad \rightarrow \text{EDO } 2^{\circ} \text{ orden en } y$$

$$\Rightarrow \hat{U}(\omega,y) = A(\omega)\, e^{-y\omega} + B(\omega)\, e^{y\omega}, \quad y>0.$$

Como U debe ser acotada para $y>0$, $\omega \in \mathbb{R}$:

$A(\omega) = 0$ si $\omega < 0$

$B(\omega) = 0$ si $\omega > 0$

$$\Rightarrow \hat{U}(\omega,y) = C(\omega)\, e^{-y|\omega|}$$

Con $y=0$: $\hat{U}(\omega,0) = \int_{-\infty}^{\infty} u(x,0)\, e^{-i\omega x}\, dx = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \hat{f}(\omega)$

$$C(\omega) = \hat{f}(\omega)$$

$$\boxed{\hat{U}(\omega,y) = \hat{f}(\omega) \cdot e^{-y|\omega|}}$$

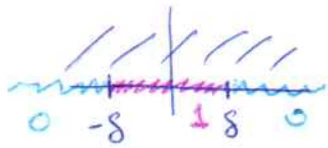
como $e^{-y|\omega|} = \hat{g}(\omega,y)$ con $g(x,y) = \frac{1}{\pi} \frac{y}{y^2+x^2}$

Convolución : $u(x,y) = f(x) * g(x,y) = \int_{-\infty}^{\infty} f(z)\, g(x-z, y)\, dz$

$$u(x,y) = \int_{-\infty}^{\infty} f(z) \cdot \frac{1}{\pi} \frac{y}{(y^2+(x-z)^2)}\, dz$$

$$= \int_{-\infty}^{\infty} f(x-z) \cdot \frac{1}{\pi} \frac{y}{y^2+z^2}\, dz$$

Ejemplo: $f(x) = \begin{cases} 1 & |x| < \delta \\ 0 & |x| > \delta . \end{cases}$



$$u(x,y) = \int_{x-\delta}^{x+\delta} \frac{1}{\pi} \frac{y}{y^2+z^2}\, dz = \frac{1}{\pi} \operatorname{arctg}\left(\frac{z}{y}\right) \Big|_{x-\delta}^{x+\delta} =$$

$$u(x,y) = \frac{1}{\pi}\left[\operatorname{arctg}\left(\frac{x+\delta}{y}\right) - \operatorname{arctg}\left(\frac{x-\delta}{y}\right)\right]$$